# UCH1603 Process Dynamics and Control

## Second Order Systems

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# Second Order System

Second order system is also called quadratic lag system. The dynamics of the system in time-domain is given by a second order differential equation, as below:

$$\tau^2 \frac{d^2Y}{dt^2} + 2\zeta\tau \frac{dY}{dt} + y = K_p X(t)$$

The corresponding transfer function is given by

$$G(s) = \frac{Y(s)}{X(s)} = \frac{K_p}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Note: this has to be 1 !

$$\begin{aligned} K_p &= \text{gain} \\ \tau &= \text{natural period of oscillation} \\ \zeta &= \text{damping factor (zeta)} \end{aligned}$$

Note: $\tau$, $\zeta > 0$. They can't take $-$ve values.

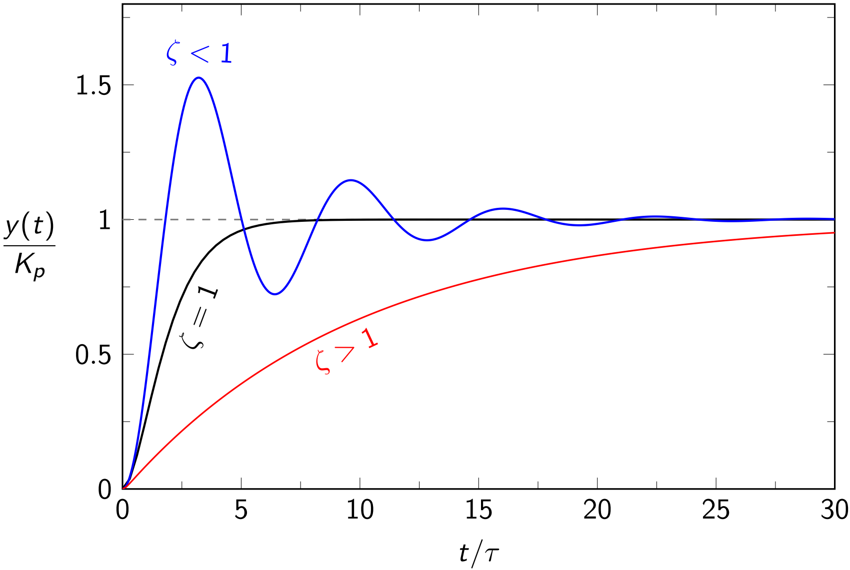
# Damping Factor

The characteristic equation of second order system is given by

$$\tau^2 s^2 + 2\zeta\tau s + 1 = 0$$

| | |
|---|---|
| If $\zeta < 1$ | underdamped system, roots are complex |
| $\zeta = 1$ | critically damped system, real and equal roots |
| $\zeta > 1$ | overdamped system, roots are real |
| $\zeta = 0$ | undamped system, complex roots with zero real part |

# Response of Second-order system to Step Input



$\zeta > 1$:

$$Y(t) = K_p \left[ 1 - e^{-\zeta t/\tau} \left( \cosh \left\{ \sqrt{\zeta^2 - 1}\, \frac{t}{\tau} \right\} + \frac{\zeta}{\sqrt{\zeta^2 - 1}} \sinh \left\{ \sqrt{\zeta^2 - 1}\, \frac{t}{\tau} \right\} \right) \right]$$

Instead of the above big formula, we shall make use of the following:

$$Y(t) = K_p \left( 1 + \frac{\tau_1 e^{-t/\tau_1}}{\tau_2 - \tau_1} - \frac{\tau_2 e^{-t/\tau_2}}{\tau_2 - \tau_1} \right)$$

from writing the transfer function as:

$$G(s) = \frac{K_p}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

# Overdamped System

- The system response resembles a little the response of a first-order system to a unit step input. But when compared to a first-order response we notice that the system initially delays to respond and then its response is rather sluggish.
- Overdamped responses are the responses of multi-capacity processes, which result from the combination of first order systems in series.

$\zeta = 1$:

$$Y(t) = K_p \left[ 1 - \left( 1 + \frac{t}{\tau} \right) e^{-t/\tau} \right]$$

- Critical damping approaches its ultimate value faster than does an overdamped system.

$0 < \zeta < 1$:

$$Y(t) = K_p \left[ 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin(\omega t + \phi) \right]$$

where

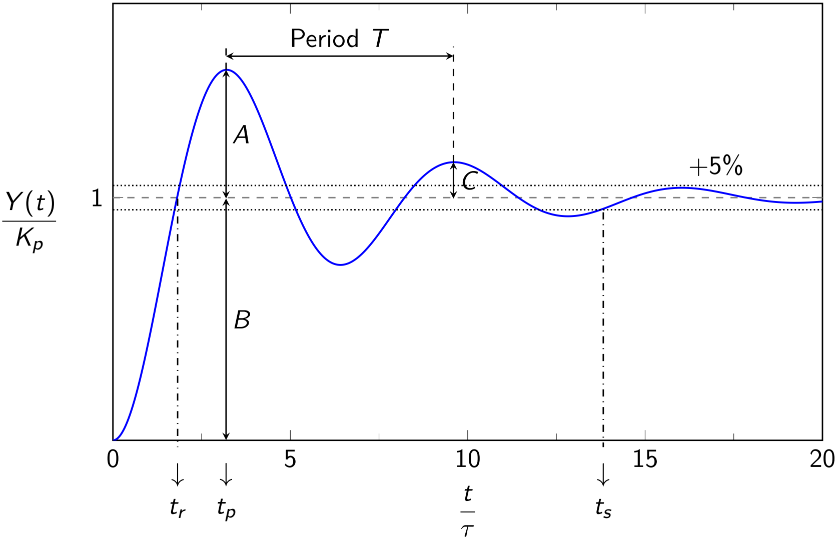
$$\omega = \frac{\sqrt{1-\zeta^2}}{\tau} \qquad \text{and} \qquad \phi = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

- Although the response is initially faster and reaches its ultimate value quickly, it does not stay there, but it starts oscillating with progressively decreasing amplitude.
- The oscillatory behavior becomes more pronounced with smaller values of the damping factor, $\zeta$.
- Almost all the underdamped responses in a chemical plant are caused by the interaction of the controllers, with the process unit they control.

# Response of Second Order System to Unit Step Input

- If $\zeta = 0$, then such a second-order system is marginally stable in that the response is of constant amplitude in time. This is the undamped case.
- If $\zeta < 0$, then such a second-order system is unstable and the response grows in time without bound.

# Characteristics of an Underdamped Response



# Characteristics of an Underdamped Response (contd..)

This is the most commonly exhibited behavior with second order systems.

(i) Rise time ($t_r$): It is the time the process output takes to first reach the new steady-state value.

(ii) Peak time ($t_p$): It is the time for the first peak to appear from the start of response.

$$t_p = \frac{\pi\tau}{\sqrt{1-\zeta^2}}$$

(iii) Overshoot: It's about how much the response exceeds its ultimate value.

$$\text{Overshoot} \ = \frac{A}{B} = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

(iv) Decay ratio: It's the ratio of successive peaks of the response.

$$\text{Decay ratio} \ = \frac{C}{A} = (\text{Overshoot})^2 = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

# Characteristics of an Underdamped Response (contd..)

(v) Period of oscillation ($T$): It is the time elapsed between two successive peaks (or two successive valleys) of the response.

$$T = \frac{2\pi\tau}{\sqrt{1-\zeta^2}}$$

and

$$\omega = 2\pi f = \frac{2\pi}{T}$$

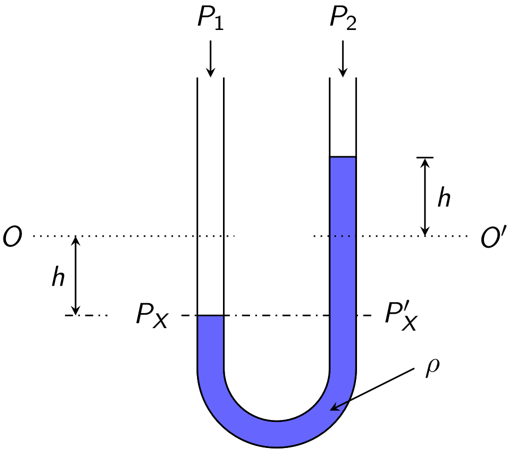
Natural cyclical period of oscillation ($T_n$) is the period of oscillation at $\zeta = 0$. Hence,

$$T_n = 2\pi\tau$$

(vi) Settling time ($t_s$): It is also known as response time. It is defined as the time required for the response to come within $\pm 5\%$ (or $\pm 2\%$) of its ultimate value and remain there. It's value is given as:

$$t_s = \frac{3\tau}{\zeta} \quad (5\% \text{ criterion}) \qquad\qquad t_s = \frac{4\tau}{\zeta} \quad (2\% \text{ criterion})$$

# Dynamics of U-tube Manometer



# Dynamics of U-tube Manometer (contd..)

Force balance at plane $XX$':

$$P_1A - P_2A - \rho g(2h)A - \left( \begin{array}{c} \text{force due to} \\ \text{fluid friction} \end{array} \right) = m\frac{dv}{dt} \qquad (1)$$

Mass of manometric fluid $= m = \rho AL$.
Velocity of fluid $= v = \dfrac{dh}{dt}$. Hence,

$$\frac{dv}{dt} = \frac{d^2h}{dt^2}$$

Force due to friction $= \Delta P_f A$.
Assuming laminar flow,

$$\Delta P_f = \frac{2fL\rho v^2}{D} = \frac{32\mu Lv}{D^2} = \frac{8\mu L}{R^2}\frac{dh}{dt}$$

# Dynamics of U-tube Manometer (contd..)

Substituting for the known quantities, in Eqn.(1), we get

$$P_1 A - P_2 A - \rho g(2h)A - \frac{8\mu LA}{R^2}\frac{dh}{dt} = \rho AL\frac{d^2h}{dt^2}$$

Dividing by $2\rho gA$ throughout, and rearranging, we get

$$\frac{L}{2g}\frac{d^2h}{dt^2} + \frac{4\mu L}{\rho g R^2}\frac{dh}{dt} + h = \frac{P_1 - P_2}{2\rho g}$$

i.e.,

$$\tau^2\frac{d^2h}{dt^2} + 2\tau\zeta\frac{dh}{dt} + h = K_p \Delta P$$

where

$$\tau^2 = \frac{L}{2g} \qquad 2\tau\zeta = \frac{4\mu L}{\rho g R^2} \qquad K_p = \frac{1}{2\rho g}$$