UCH1603 Process Dynamics and Control

Frequency Response

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Introduction

- For the stability analysis and design of feedback controllers, frequency response analysis is very useful.
- When a linear system is subjected to a sinusoidal input, its ultimate response (after a long time) is also a sustained sinusoidal wave (will be proved...). This constitutes the basis of frequency response analysis.
- With frequency response analysis we are interested primarily in determining how the features of the output sinusoidal wave (amplitude, phase shift) change with the frequency of the input sinusoid.

Response of a First Order System to a Sinusoidal Input

For a first order system,

$$G_p(s) = \frac{Y(s)}{X(s)} = \frac{K_p}{\tau_p s + 1}$$

Let $X(t)$ be a sinusoidal input with amplitude A and frequency ω.

$$X(t) = A\sin(\omega t)$$

Then,

$$X(s) = \frac{A\omega}{s^2 + \omega^2}$$

Hence,

$$Y(s) = \frac{K_p}{\tau_p s + 1}\,\frac{A\omega}{s^2 + \omega^2} = \frac{C_1}{s + 1/\tau_p} + \frac{C_2}{s + j\omega} + \frac{C_3}{s - j\omega}$$

First Order System to Sinusoidal Input (contd..)

Evaluating the coefficients C_1, C_2, C_3, and taking inverse Laplace transform, we get

$$Y(t) = \frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} e^{-t/\tau_p} - \frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} \cos(\omega t) + \frac{K_p A}{\tau_p^2 \omega^2 + 1} \sin(\omega t)$$

As $t \to \infty$, $e^{-t/\tau_p} \to 0$. Hence,

$$Y_{ss}(t) = -\frac{K_p A \omega \tau_p}{\tau_p^2 \omega^2 + 1} \cos(\omega t) + \frac{K_p A}{\tau_p^2 \omega^2 + 1} \sin(\omega t)$$

Using the following trigonometric identity in the above:

$$a_1 \cos b + a_2 \sin b = a_3 \sin(b + \phi)$$

where

$$a_3 = \sqrt{a_1^2 + a_2^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{a_1}{a_2}\right)$$

First Order System to Sinusoidal Input (contd..)

$$Y_{ss}(t) = \frac{AK_p}{\sqrt{\tau_p^2\omega^2 + 1}} \sin(\omega t + \phi)$$

where

$$\phi = \tan^{-1}(-\omega\tau_p)$$

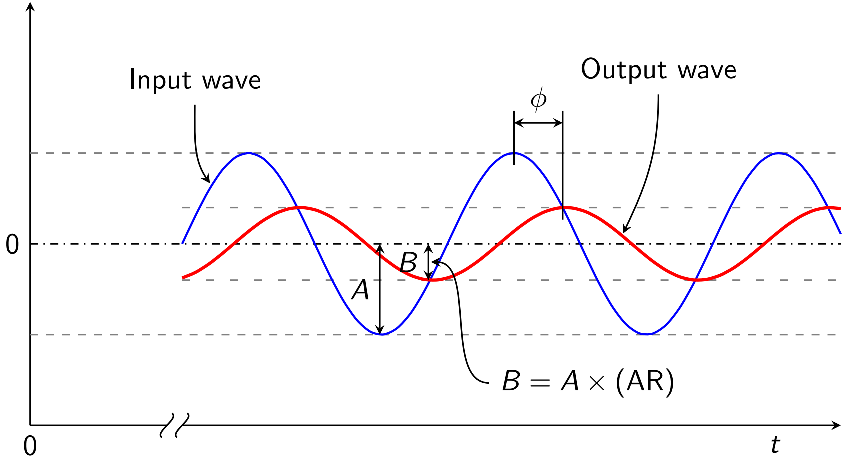
First Order System to Sinusoidal Input (contd..)

- The ultimate response (also referred to as steady state) of a first order system to a sinusoidal input is also a sinusoidal wave with the same frequency ω.
- The ration of output amplitude to input amplitude is called the amplitude ratio (AR) and is a function of frequency.

$$\mathrm{AR} = \frac{K_p}{\sqrt{\tau_p^2\omega^2 + 1}}$$

- The output wave lags behind the input wave (phase lag) by an angle $|\phi|$, which is also a function of ω.

First Order System to Sinusoidal Input (contd..)



Complex Number

Consider a complex number z defined by

$$z = x + jy$$

The modulus or absolute value of z is represented by $|z|$ and defined by

$$|z| = \sqrt{x^2 + y^2}$$

The phase angle or argument of z is represented by $\angle z$ and defined by

$$\angle z = \tan^{-1}\left(\frac{y}{x}\right)$$

Amplitude Ratio and Phase Shift of First Order System

For a first order system,

$$G(s) = \frac{K_p}{\tau_p s + 1}$$

Put $s = j\omega$.

$$\begin{aligned} G(j\omega) &= \frac{K_p}{\tau_p j\omega + 1} \\ &= \frac{K_p}{1 + j\omega\tau_p} = \frac{K_p}{(1 + j\omega\tau_p)} \frac{(1 - j\omega\tau_p)}{(1 - j\omega\tau_p)} \\ &= \frac{K_p}{\tau_p^2\omega^2 + 1} - j\frac{K_p\tau_p\omega}{\tau_p^2\omega^2 + 1} \qquad (1) \end{aligned}$$

Amplitude Ratio and Phase Shift of First Order System (contd..)

For Eqn.(1),

$$
\begin{aligned}
\text{Modulus of } G(j\omega) &= \sqrt{\left(\frac{K_p}{\tau_p^2\omega^2+1}\right)^2 + \left(\frac{-K_p\omega\tau_p}{\tau_p^2\omega^2+1}\right)^2} \\
&= \sqrt{\frac{K_p^2 + K_p^2\omega^2\tau_p^2}{(\tau_p^2+1)^2}} \\
&= \frac{K_p}{\sqrt{\tau_p^2\omega^2+1}} = \text{ Amplitude Ratio} = \text{AR}
\end{aligned}
$$

Amplitude Ratio and Phase Lag of First Order System (contd..)

For Eqn.(1),

$$\text{Argument of } G(j\omega) = \tan^{-1}(-\omega\tau_p) = \text{Phase Shift} = \phi$$

This is a phase lag since $\phi < 0$.

Frequency Response of General Linear System

For a linear system with transfer function $G(s) = Y(s)/X(s)$, for input $X(t) = A\sin(\omega t)$, output at steady state (i.e., ultimate response) is given by

$$Y_{ss}(t) = A|G(j\omega)|\sin(\omega t + \phi)$$

- The ultimate response as $t \to \infty$ is sinusoidal with frequency ω.
- The amplitude ratio is

$$\text{AR} = \frac{A|G(j\omega)|}{A} = |G(j\omega)|$$

- The output sinusoidal wave has been shifted by the angle

$$\phi = \text{argument of } G(j\omega)$$

Frequency Response of Pure Capacitive Process

For a pure capacitive process

$$G(s) = \frac{K_p}{s}$$

Put $s = j\omega$

$$G(j\omega) = \frac{K_p}{j\omega} = \frac{K_p}{j\omega}\frac{j\omega}{j\omega} = 0 - j\frac{K_p}{\omega}$$

- The amplitude ratio is

$$\text{AR} = |G(j\omega)| = \sqrt{0^2 + \left(\frac{-K_p}{\omega}\right)^2} = \frac{K_p}{\omega}$$

- The phase shift is

$$\phi = \tan^{-1}\left(\frac{-Kp/\omega}{0}\right) = \tan^{-1}(-\infty) = -90^\circ$$

Frequency Response of Second Order System

For a second order system

$$G(s) = \frac{K_p}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Put $s = j\omega$

$$\begin{aligned}
G(j\omega) &= \frac{K_p}{\tau^2 (j\omega)^2 + 2\zeta\tau j\omega + 1} \\
&= \frac{K_p}{(-\tau^2\omega^2 + 1) + j2\zeta\tau\omega} \\
&= \frac{K_p}{(-\tau^2\omega^2 + 1) + j2\zeta\tau\omega} \frac{(-\tau^2\omega^2 + 1) - j2\zeta\tau\omega}{(-\tau^2\omega^2 + 1) - j2\zeta\tau\omega} \\
&= \frac{K_p(1 - \tau^2\omega^2)}{(1 - \tau^2\omega^2)^2 + (2\zeta\tau\omega)^2} - j\frac{K_p \cdot 2\zeta\tau\omega}{(1 - \tau^2\omega^2)^2 + (2\zeta\tau\omega)^2}
\end{aligned}$$

Frequency Response of Second Order System (contd..)

Amplitude ratio:

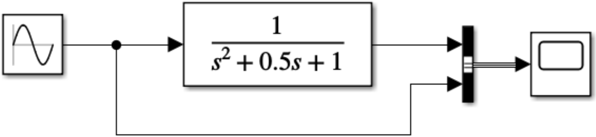
$$\text{AR} = |G(j\omega)| = \frac{K_p}{\sqrt{(1-\tau^2\omega^2)^2 + (2\zeta\tau\omega)^2}}$$

Phase shift:

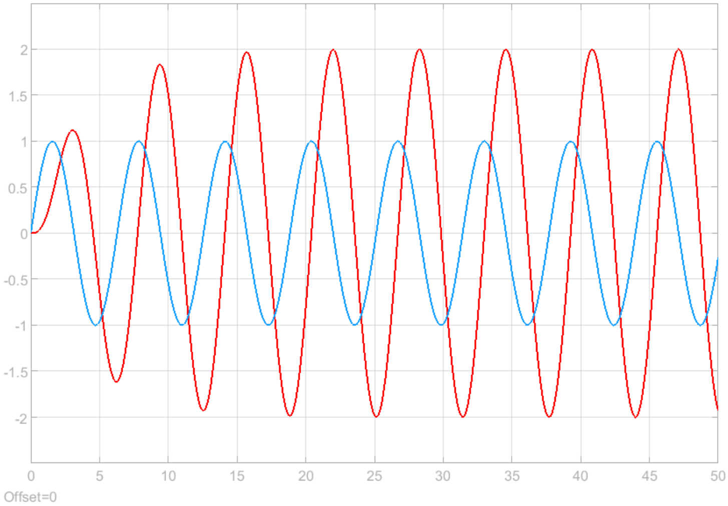
$$\phi = \text{argument of } G(j\omega) = \tan^{-1}\left(-\frac{2\zeta\tau\omega}{1-\tau^2\omega^2}\right)$$

This is a phase lag since $\phi < 0$.

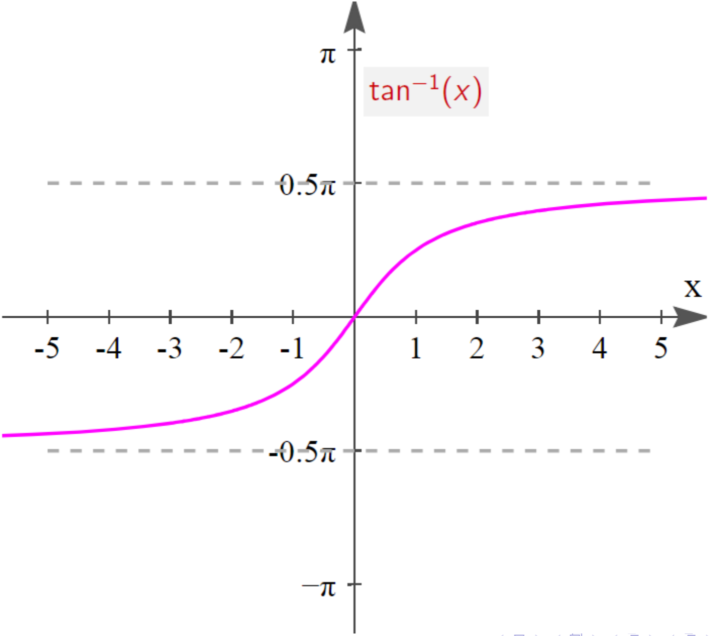
Frequency Response of Second Order System (contd..)



Frequency Response of Second Order System (contd..)



Graph of $\tan^{-1}(x)$



AR and ϕ Various Systems for Sinusoidal Input

System	Transfer function	Amplitude ratio (AR)	Phase shift (ϕ)
First order	$\frac{K_p}{\tau_p s + 1}$	$\frac{K_p}{\sqrt{\tau_p^2 \omega^2 + 1}}$	$\tan^{-1}(-\omega \tau_p)$
Pure capacitive	$\frac{K_p}{s}$	$\frac{K_p}{\omega}$	-90^o
Second order	$\frac{K_p}{\tau^2 s^2 + 2\zeta \tau s + 1}$	$\frac{K_p}{\sqrt{(1 - \tau^2 \omega^2)^2 + (2\zeta \tau \omega)^2}}$	$\tan^{-1}\left(-\frac{2\zeta \tau \omega}{1 - \tau^2 \omega^2}\right)$
Dead time	$e^{-\tau_d s}$	1	$-\tau_d \omega$
Systems in series	$G_1(s) G_2(s) \cdots G_N(s)$	$(\text{AR})_1 (\text{AR})_2 \cdots (\text{AR})_N$	$\phi_1 + \phi_2 + \cdots + \phi_N$

AR and ϕ Various Systems for Sinusoidal Input (contd..)

System	Transfer function	Amplitude ratio (AR)	Phase shift (ϕ)
Proportional controller	K_c	K_c	0
PI controller	$K_c\left(1+\frac{1}{\tau_I s}\right)$	$K_c\sqrt{1+\frac{1}{(\omega\tau_I)^2}}$	$\tan^{-1}\left(\frac{-1}{\omega\tau_I}\right)$
PD controller	$K_c(1+\tau_D s)$	$K_c\sqrt{1+\tau_D^2\omega^2}$	$\tan^{-1}(\tau_D\omega)$
PID controller	$K_c\left(1+\frac{1}{\tau_I s}+\tau_D s\right)$	$K_c\sqrt{\left(\tau_D\omega-\frac{1}{\tau_I\omega}\right)^2+1}$	$\tan^{-1}\left(\tau_D\omega-\frac{1}{\tau_I\omega}\right)$