Heat Transfer

Conduction - Fins

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Objectives

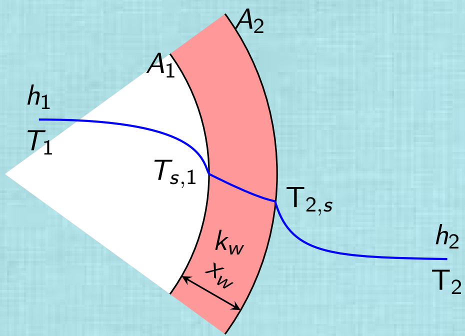
- To give an idea about increasing the rate of heat transfer by extending the heat transfer surfaces (i.e., fins).
- To derive the equation for steady state heat conduction through simple extended surface geometries
- To estimate the performance parameters of fins.

Outcome

- To understand the need and usage of fins.
- To calculate the heat transfer augmentation by simple fins.

Overall Heat Transfer Coefficient (U)

$$Q = \frac{\Delta T}{R} = UA\Delta T = U_1 A_1 \Delta T = U_2 A_2 \Delta T$$



$$T_1 - T_{1,s} = \frac{Q}{h_1 A_1} \qquad T_{1,s} - T_{2,s} = \frac{x_w Q}{k_w A_m} \qquad T_{2,s} - T_2 = \frac{Q}{h_2 A_2}$$

Adding the numerator and denominator separately, we get

$$T_1 - T_2 = \Delta T = Q \left[\frac{1}{h_1 A_1} + \frac{x_w}{k_w A_m} + \frac{1}{h_2 A_2} \right]$$

From the relation $Q = U_1 A_1 \Delta T$, we get

$$\frac{Q}{U_1 A_1} = Q \left[\frac{1}{h_1 A_1} + \frac{x_w}{k_w A_m} + \frac{1}{h_2 A_2} \right]$$

i.e.,

$$\frac{1}{U_1 A_1} = \frac{1}{h_1 A_1} + \frac{x_w}{k_w} \frac{A_1}{A_m} + \frac{1}{h_2 A_2} = \frac{1}{U_2 A_2}$$

Overall Heat Transfer Coefficient (contd..)

For highly conducting and / or thin-walled tubes, we can neglect the conductive resistance part, and hence:

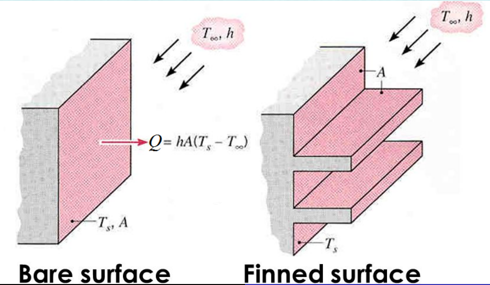
$$\frac{1}{U_1 A_1} = \frac{1}{h_1 A_1} + \frac{1}{h_2 A_2}$$

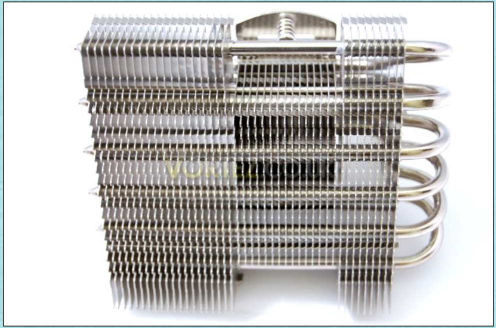
$$Q = hA(T_s - T_\infty)$$

The rate of convective heat transfer from a surface at T_s can be increased by two methods:

- increasing the convective heat transfer coefficient, h
- increasing the surface area, A

Increasing the convective heat transfer coefficient may not be practical and/or adequate. An increase in surface area by attaching extended surfaces called fins to the surface is more convenient.



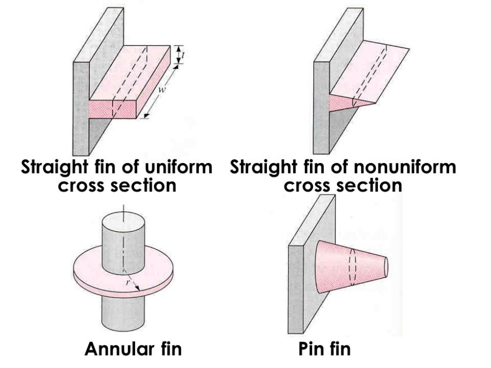


Examples of Extended Surfaces (Fins):

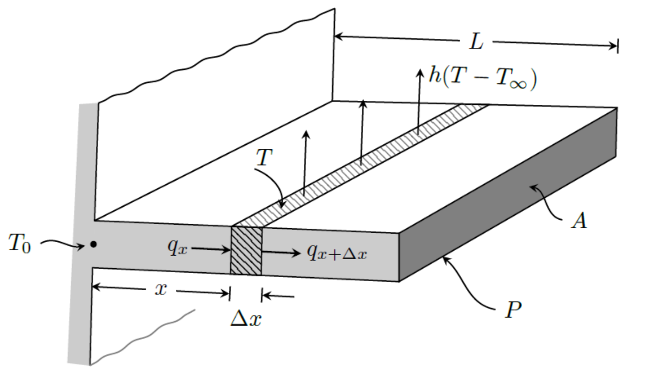
- Thin rods on condenser in back of refrigerator
- Honeycomb surface of a car radiator
- Corrugated surface of a motorcycle engine

Fins

Finned surfaces are commonly used in practice to enhance heat transfer. In the analysis of fins, we consider steady operation with no heat generation in the fin. We also assume that the convection heat transfer coefficient, h to be constant and uniform over the entire surface of the fin.



One Dimensional Fin Equation



$$\text{Heat in at } x \text{ by conduction} = \text{Heat out by conduction at } (x + \Delta x)$$
$$+ \text{ Heat out by convection}$$

i.e.,

$$\text{Net heat in by conduction} = \text{Heat out by convection}$$

i.e.,

$$(Aq)|_x - (Aq)|_{x+\Delta x} = hP\Delta x(T - T_\infty) \tag{1}$$

Dividing throughout by Δx, and, from the definition of derivative, (for $\Delta x \to 0$)

$$-\frac{((Aq)|_{x+\Delta x} - (Aq)|_x)}{\Delta x} = -\frac{d(Aq)}{dx}$$

One Dimensional Fin Equation (contd..)

From Fourier's law of conduction, $q = -k\dfrac{dT}{dx}$. Therefore,

$$-\frac{d(Aq)}{dx} = Ak\frac{d^2T}{dx^2}$$

Using this in Eqn.(1), we get

$$Ak\frac{d^2T}{dx^2} = hP(T - T_\infty)$$

i.e.,

$$\frac{d^2T}{dx^2} - \frac{hP}{Ak}(T - T_\infty) = 0$$

or,

$$\boxed{\frac{d^2\theta}{dx^2} - m^2\theta = 0}$$

where

$$m^2 = \frac{hP}{Ak} \qquad \text{and} \qquad \theta = (T - T_\infty)$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

The above equation is a linear homogeneous, second-order ordinary differential equation. The solution of which is given by

$$\theta = C_1 e^{-mx} + C_2 e^{mx}$$

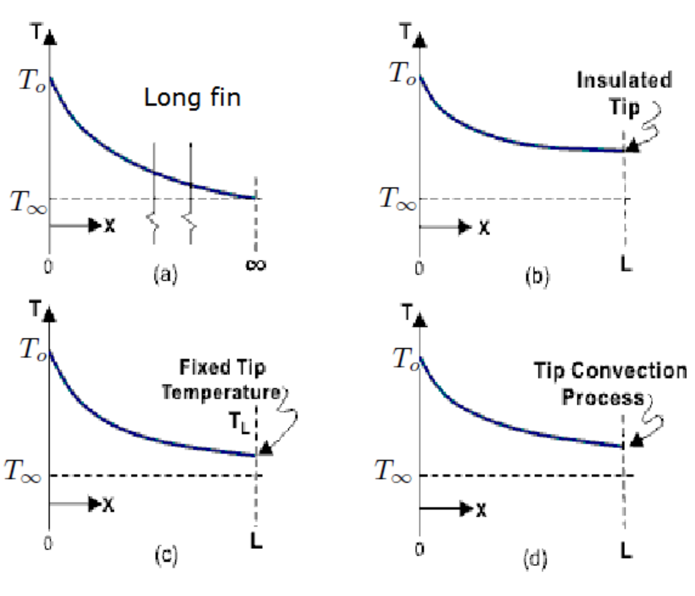
Or,

$$\theta = C_1' \cosh mx + C_2' \sinh mx$$

Or,

$$\theta = C_1'' \cosh m(L - x) + C_2'' \sinh m(L - x)$$

Fin - Boundary Conditions



$$\theta = C_1 e^{-mx} + C_2 e^{mx} \tag{1}$$

For a sufficiently long fin, it is reasonable to assume that the temperature of the fin tip approaches T_∞.
Boundary Conditions (BC):

$$\begin{array}{llll} \text{BC-1} & \theta = \theta_o = (T_o - T_\infty) & \text{at} & x = 0 \\ \text{BC-2} & \theta = 0 & \text{at} & x \to \infty \end{array}$$

Application of BC-2 to Eqn.(1) gives

$$\begin{aligned} 0 &= C_1 e^{-\infty m} + C_2 e^{\infty m} \\ &= 0 + C_2 C_3 \\ \implies \quad C_2 &= 0 \end{aligned}$$

And, from BC-1 to Eqn.(1) we get

$$\theta_o = C_1 e^{-0m} \quad \implies \quad C_1 = \theta_o$$

Long Fin (contd..)

Hence,

$$\theta = \theta_o e^{-mx} \qquad \Longrightarrow \qquad \frac{\theta}{\theta_o} = e^{-mx}$$

i.e.,

$$\frac{\theta}{\theta_o} = \frac{T - T_\infty}{T_o - T_\infty} = e^{-mx}$$

Heat flow through the fin (Q) is given by

$$Q = \int_{x=0}^{L} hP\theta dx$$

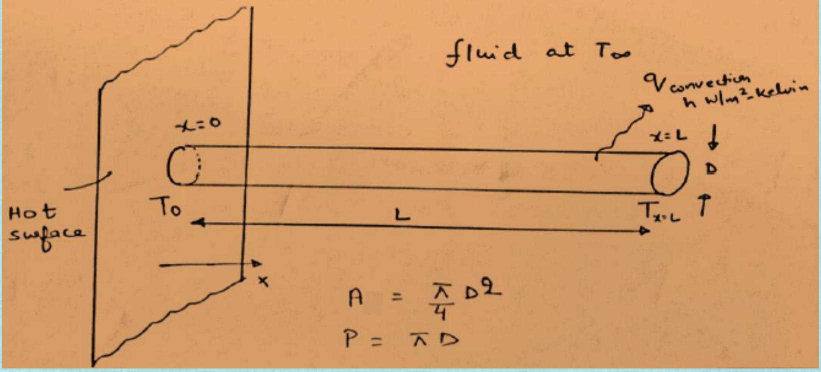
or,

$$Q = -Ak \left. \frac{d\theta}{dx} \right|_{x=0}$$

From any of the above two equations, we get $Q = Akm\theta_o$.
i..e,

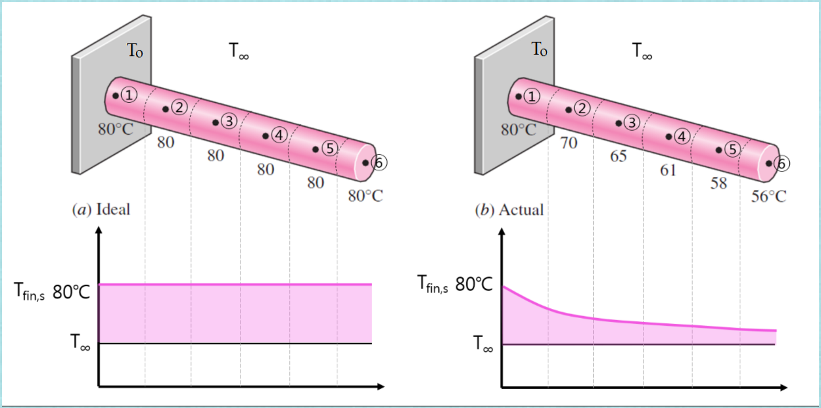
$$\boxed{Q = \theta_o \sqrt{PhkA}}$$

Pin Fin



$A = \frac{\pi}{4} D^2$

$P = \pi D$


To
T∞
1
2
3
4
5
6
80°C
80
80
80
80
80°C
(a) Ideal
Tfin,s 80°C
T∞
To
T∞
1
2
3
4
5
6
80°C
70
65
61
58
56°C
(b) Actual
Tfin,s 80°C
T∞

Fin Efficiency (contd..)

Temperature of a fin gradually drops along the length. In the limiting case of zero thermal resistance ($k \to \infty$), the temperature of the fin will be uniform at the base value of T_o. The heat transfer from the fin will be maximized in this case:

$$Q_{\text{fin,max}} = hA_{\text{fin}}(T_o - T_\infty)$$

Fin efficiency (η_{fin}) can be defined as:

$$\eta_{\text{fin}} = \frac{Q_{\text{fin}}}{Q_{\text{fin,max}}} = \frac{\text{actual heat transfer rate from the fin}}{\text{ideal heat transfer rate from the fin}}$$

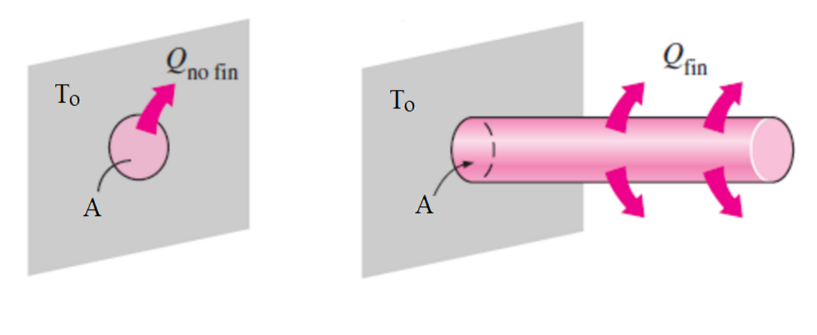
(if the entire fin were at base temperature)

Fin efficiency decreases with increasing fin length because of decrease in fin temperature with length.

For the long-fin

$$\eta_{\text{fin}} = \frac{Q_{\text{fin}}}{Q_{\text{fin,max}}} = \frac{\sqrt{PhkA}(T_o - T_\infty)}{hA_f(T_o - T_\infty)} = \frac{\sqrt{PhkA}}{hPL} = \frac{1}{L}\sqrt{\frac{kA}{hP}} = \frac{1}{mL}$$

Fin Effectiveness



Fin Effectiveness (contd..)

The performance of fins is judged on the basis of the enhancement in heat transfer relative to the no-fin case, and expressed in terms of the fin effectiveness:

$$\varepsilon_{\text{fin}} = \frac{Q_{\text{fin}}}{Q_{\text{no fin}}} = \frac{\text{heat transfer rate with fin}}{\text{heat transfer rate without fin}}$$

$$\varepsilon_{\text{fin}} = \begin{cases} < 1 & \text{fin act as insulation} \\ = 1 & \text{fin does not affect heat transfer} \\ > 1 & \text{fin enhances heat transfer} \end{cases}$$

$$\eta_{\text{fin}} < 1 \qquad \text{but} \qquad \varepsilon_{\text{fin}} > 1$$

Fin Effectiveness (contd..)

For the long-fin

$$\varepsilon_{\text{fin}} = \frac{Q_{\text{fin}}}{Q_{\text{no fin}}} = \frac{\sqrt{PhkA}(T_o - T_\infty)}{hA(T_o - T_\infty)} = \frac{\sqrt{PhkA}}{hA} = \sqrt{\frac{kP}{hA}}$$

and,

$$\frac{\varepsilon_{\text{fin}}}{\eta_{\text{fin}}} = \frac{PL}{A}$$

Effectiveness of fin must be greater than 2; otherwise don't use the fin.

Fins - Requirements

- Fins are generally used where convective heat transfer coefficient (h) values are relatively low. i.e., when air or gas is the medium and heat transfer is by natural convection.
- Fin material should be of highly conductive materials.
- Lateral surface area of the fin should be as high as possible.
- The efficiency of most fins used in practice is above 90%.

Example 1: Effect of Diameter and Thermal Conductivity on Heat Transfer through Fin

A long, circular aluminium rod attached at one end to the heated wall and transfers heat through convection to a cold fluid.

(a) If the diameter of the rod is triples, by how much would the rate of heat removal change?

(b) If a copper rod of the same diameter is used in place of aluminium, by how much would the rate of heat removal change?
aluminum: $k = 240$ W/(m.K); copper: $k = 400$ W/(m.K)

Solution:

For long-fin,

$$Q = \theta_o \sqrt{PhkA}$$

For cyldrical fin,

$$P = \pi D \qquad \text{and} \qquad A = \frac{\pi}{4} D^2 \qquad \Longrightarrow \qquad PA = \frac{\pi}{4} D^3$$

Therefore,

$$Q \propto \sqrt{kD^3}$$

(a) $D_2/D_1 = 3$. Therefore,

$$\frac{Q_2}{Q_1} = \frac{\sqrt{D_2^3}}{\sqrt{D_1^3}} = \sqrt{\left(\frac{D_2}{D_1}\right)^3} = \sqrt{3^3} = 5.2$$

i.e., there is a 520% increase in heat transfer.

(b) $k_2/k_1 = 400/240 = 1.667$. Therefore,

$$\frac{Q_2}{Q_1} = \sqrt{\frac{k_2}{k_1}} = \sqrt{1.667} = 1.29$$

i.e., there is a 29% increase in heat transfer.

Fin with Negligible Heat Transfer at the Tip

$$\theta = C_1 \cosh m(L-x) + C_2 \sinh m(L-x) \qquad (1)$$

The heat transfer area at the fin tip is generally small compared with the lateral area of the fin for heat transfer. For such situations, the heat loss from the fin tip is negligible compared with that from lateral surfaces, and the boundary condition at the tip characterizing this situation is taken as $d\theta/dx = 0$ at $x = L$.
Boundary Conditions (BC):

$$\theta = \theta_o = (T_o - T_\infty) \qquad \text{at} \quad x = 0$$

$$\frac{d\theta}{dx} = 0 \qquad \text{at} \quad x = L$$

Differentiating Eqn.(1),

$$\frac{d\theta}{dx} = -mC_1 \sinh m(L-x) - mC_2 \cosh m(L-x)$$

Using B.C 2 on the above,

$$\begin{aligned}
0 &= -mC_1 \sin hm0 - mC_2 \cosh m0 \\
00 &- C_2 \qquad (\because \sinh 0 = 0 \quad \text{and} \quad \cosh 0 = 1) \\
\implies \quad C_2 &= 0
\end{aligned}$$

Therefore, Eqn.(1) becomes,

$$\theta = C_1 \cosh m(L - x)$$

From B.C 1, at $x = 0$, $\theta = \theta_o$. Using this in above equation,

$$\theta_o = C_1 \cosh mL \qquad \implies \qquad C_1 = \frac{\theta_o}{\cosh mL}$$

Hence, we get

$$\boxed{\frac{\theta}{\theta_o} = \frac{\cosh m(L - x)}{\cosh mL}}$$

Fin with Negligible Heat Transfer at the Tip (contd..)

Taking derivative of the temperature distribution equation, and at $x = 0$ we get

$$\left.\frac{d\theta}{dx}\right|_{x=0} = -m\theta_o \frac{\sinh m(L-0)}{\cosh mL} = -\theta_o m \tanh mL$$

Heat transfer through the fin is given by

$$Q = -\,kA\left.\frac{d\theta}{dx}\right|_{x=0} = kA\theta_o m \tanh mL = \theta_o \sqrt{PhkA} \tanh mL$$

$$\theta = C_1 e^{-mx} + C_2 e^{mx} \tag{1}$$

i.e., at $x = 0$, $T = T_o$; and, at $x = L$, $T = T_L$.
B.C.:

$$\theta = \theta_o \qquad \text{at } x = 0$$
$$\theta = \theta_L \qquad \text{at } x = L$$

Using B.C 1 in Eqn.(1), we get

$$\theta_o = C_1 + C_2 \qquad \Longrightarrow \qquad C_1 = \theta_o - C_2 \tag{2}$$

And, using B.C 2 in Eqn.(1), we get

$$\theta_L = C_1 e^{-mL} + C_2 e^{mL} \tag{3}$$

Using Eqn.(2) in Eqn.(3), we get

$$\begin{aligned}\theta_L &= (\theta_o - C_2)e^{-mL} + C_2 e^{mL} \\ &= \theta_o e^{-mL} - C_2 e^{-mL} + C_2 e^{mL} \\ &= \theta_o e^{-mL} + C_2(e^{mL} - e^{-mL}) \\ \Longrightarrow \quad C_2 &= \frac{\theta_L - \theta_o e^{-mL}}{e^{mL} - e^{-mL}} \end{aligned} \qquad (4)$$

Using Eqn.(4) in Eqn.(2), we get

$$\begin{aligned} C_1 &= \theta_o - \frac{\theta_L - \theta_o e^{-mL}}{e^{mL} - e^{-mL}} \\ &= \frac{\theta_o(e^{mL} - e^{-mL}) - \theta_L + \theta_o e^{-mL}}{e^{mL} - e^{-mL}} \\ \Longrightarrow \quad C_1 &= \frac{\theta_o e^{mL} - \theta_L}{e^{mL} - e^{-mL}} \end{aligned} \qquad (5)$$

Substituting for C_1 and C_2 in Eqn.(1), we get

$$\theta = \frac{\theta_o e^{mL} - \theta_L}{e^{mL} - e^{-mL}} e^{-mx} + \frac{\theta_L - \theta_o e^{-mL}}{e^{mL} - e^{-mL}} e^{mx}$$

$$= \frac{\theta_o e^{mL} e^{-mx} - \theta_o e^{-mL} e^{mx} + \theta_L e^{mx} - \theta_L e^{-mx}}{e^{mL} - e^{-mL}}$$

$$= \frac{\theta_o \left(e^{m(L-x)} - e^{-m(L-x)}\right) + \theta_L (e^{mx} - e^{-mx})}{e^{mL} - e^{-mL}}$$

We know,

$$\frac{e^{mx} - e^{-mx}}{2} = \sinh mx$$

Using this in above, we get

$$\boxed{\theta = \frac{\theta_o \sinh m(L-x) + \theta_L \sinh mx}{\sinh mL}}$$

Fin with Specified Temperature at the Ends (contd..)

Rate of heat transfer through the fin at $x = 0$ is given by

$$Q_o = -kA\left.\frac{d\theta}{dx}\right|_{x=0} = -kA\left.\frac{-m\theta_o \cosh m(L-x) + m\theta_L \cosh mx}{\sinh mL}\right|_{x=0}$$

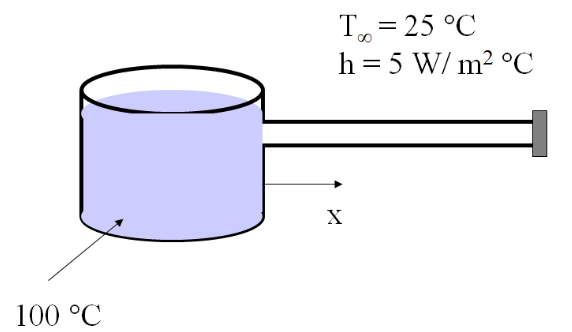
$$= kAm\theta_o \frac{\cosh mL - \theta_L/\theta_o}{\sinh mL}$$

i.e.,

$$Q_o = \theta_o\sqrt{PhkA}\frac{\cosh mL - \theta_L/\theta_o}{\sinh mL}$$

Example 1: Fin with Adiabatic Tip

An aluminum pot is used to boil water as shown below. The handle of the pot is 20 cm long, 3 cm wide, and 0.5 cm thick. The pot is exposed to room air at 25°C, and the convection coefficient is 5 W/m^2.°C. Can you touch the handle when the water is boiling? What would be the temperature near the end of the handle? (*k* for aluminum is 237 W/m.°C)



Solved Problems (contd..)

Solution:

We can model the pot handle as an extended surface. Assume that there is no heat transfer at the free end of the handle. For this case,

$$\frac{T - T_\infty}{T_o - T_\infty} = \frac{\cosh m(L - x)}{\cosh mL}$$

where

$$m = \sqrt{\frac{Ph}{kA}}$$

Here, $h = 5$ W/m^2.°C, $P = 2(W + B) = 2(0.03 + 0.005) = 0.07$ m, $k = 237$ W/m.°C, $A = WB = 0.03 \times 0.005 = 0.00015$ m^2, and $L = 0.2$ m.
Therefore,

$$m = \sqrt{\frac{Ph}{kA}} = \sqrt{\frac{0.07 \times 5}{237 \times 0.00015}} = 3.138$$

Solved Problems (contd..)

Hence,

$$\frac{T - T_\infty}{T_o - T_\infty} = \frac{\cosh m(L - x)}{\cosh mL}$$

$$\frac{T - 25}{100 - 25} = \frac{\cosh(3.138 \times (0.2 - 0.2))}{\cosh(3.138 \times 0.2)}$$

$$\implies \quad T(x = 0.2) = 87.3^\circ\mathrm{C}$$

Since T near the end is $87.3^\circ\mathrm{C}$, it is not safe to touch the end. If a stainless steel handle is used instead, what will happen? For stainless steel, the thermal conductivity $k = 15\ \mathrm{W/m.^\circ C}$ (Ans: $37.3^\circ\mathrm{C}$; safer than the previous case)