Heat Transfer

Conduction - Unsteady Heat Conduction

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Objectives

- To introduce the methodologies of solving simple unsteady heat conduction problems.

Outcome

- To obtain the equation for variation of temperature with time, using lumped system approach.
- To estimate the parameters of one dimensional unsteady heat transfer using transient-temperature charts.

Lumped Parameter Formulation

A lumped parameter formulation is an approximation which facilitates the solution of heat transfer problems. The key assumption is the neglect of temperature gradients (dT/dx) inside the body of volume V and surface area A, so that its temperature is only a function of time. However, this assumption clearly amounts to the neglect of the heat conduction processes inside the material and should be used with caution.

This approximation is valid if

$$\mathrm{Bi} = \frac{hL}{k} < 0.1$$

where $L = V/A$

Lumped Parameter Formulation (contd..)

If the only mechanism for energy exchange with the surroundings is convection through the bounding surface, the differential thermal energy balance equation (in W) becomes

$$-hA(T - T_\infty) = \rho C_P V \frac{dT}{dt}$$

This is a first order ordinary differential equation which can be solved easily. Rearranging the above,

$$\frac{dT}{dt} + \frac{hA}{\rho C_P V}(T - T_\infty) = 0$$

$$\frac{dT}{dt} + m(T - T_\infty) = 0$$

where

$$m = \frac{hA}{\rho C_P V}$$

Let $\theta = T - T_\infty$. Then, $\dfrac{d\theta}{dt} = \dfrac{dT}{dt}$. Therefore,

$$\frac{d\theta}{dt} + m\theta = 0$$

Solution of the above ODE is given by

$$\frac{d\theta}{\theta} = -mdt$$

Integrating,

$$\ln\theta = -mt + C_1$$

Initial condition: At $t = 0$, $\theta = \theta_0 = T_0 - T_\infty$. Substituting this in the above, we get

$$\ln\theta_0 = C_1$$

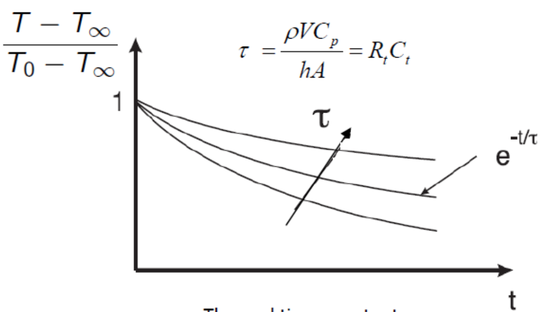
Hence,

$$\ln\theta = -mt + \ln\theta_0 \qquad \Longrightarrow \qquad \frac{\theta}{\theta_0} = e^{-mt}$$

Lumped Parameter Formulation (contd..)

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-mt}$$

$$m = \frac{hA}{\rho C_P V} = \frac{1}{\tau}$$



Thermal time constant

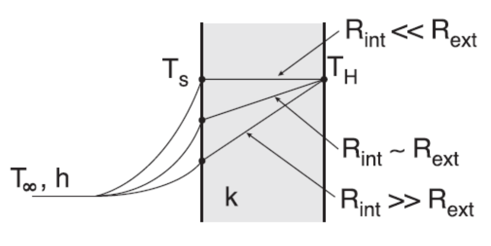
Biot Number

$$\text{Bi} = \frac{hL}{k} = \frac{\text{resistance to internal heat flow}}{\text{resistance to external heat flow}}$$

Whenever the Biot number is small, the internal temperature gradients are also small and a transient problem can be treated by the"lumped thermal capacity" approach. The lumped capacity assumption implies that the object for analysis is considered to have a single temperature.

Biot Number (contd..)

$$\frac{T_H - T_S}{T_S - T_\infty} = \frac{L/(kA)}{1/(hA)} = \frac{\text{internal resistance to H.T}}{\text{external resistance to H.T}} = \frac{hL}{k} = \text{Bi}$$



$R_{\text{int}} \ll R_{\text{ext}}$: the Biot number is small and we can conclude

$$T_H - T_S \ll T_S - T_\infty \qquad \text{and in the limit } T_H \approx T_S$$

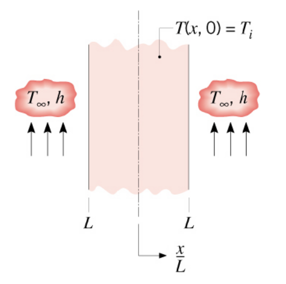
$R_{\text{ext}} \ll R_{\text{int}}$: the Biot number is large and we can conclude

$$T_S - T_\infty \ll T_H - T_S \qquad \text{and in the limit } T_S \approx T_\infty$$

A 150 micrometer diameter steel sphere of $\rho = 7,700$ kg/m^3, $C_P = 460$ J/(kg.K), $k = 25$ W/(m.K) is quenched from a temperature of 1200 K using an air jet with $h = 100$ W/(m^2.K), at room temperature ($T_\infty = 300$ K). Calculate the value of Bi for this system and, if possible, use the lumped parameter model to estimate the time it takes for the temperature of the sphere to reach 325 K.

Use of Transient Temperature Charts

If the lumped capacitance approximation can not be made, consideration must be given to spatial, as well as temporal, variations in temperature during the transient process.



For slab of of thickness $2L$, considering symmetry with respect to $x = 0$ at the midplane, with constant k, and no heat generation, we have

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in } 0 < x < L, \text{ for } t > 0$$

Boundary and Initial Conditions:

$$
\begin{aligned}
\frac{\partial T}{\partial x} &= 0 && \text{at } x = 0, \text{ for } t > 0 \\
k\frac{\partial T}{\partial x} + hT &= hT_\infty && \text{at } x = L, \text{ for } t > 0 \\
T &= T_i && \text{for } t = 0, \text{ in } 0 \leq x \leq L
\end{aligned}
$$

Dimensionless Quantities:

$$\theta = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$ dimensionless temperature

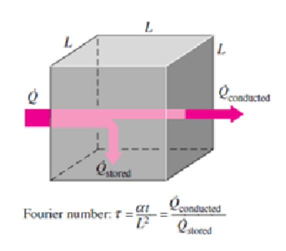
$$X = \frac{x}{L}$$ dimensionless coordinate

$$\mathrm{Bi} = \frac{hL}{k}$$ Biot number

$$\tau = \frac{\alpha t}{L^2}$$ dimensionless time, or Fourier number

$$\tau = \text{Fo} = \frac{\alpha t}{L^2} = \frac{k(1/L)L^2\,\Delta T}{\rho C_P L^3\,\Delta T/t} = \frac{\text{rate of heat conduction across } L \text{ in volume } L^3}{\text{rate of heat storage in volume } L^3}$$

Fourier number is a measure of heat conducted through a body relative to heat stored. Thus, a large value of the Fourier number indicates faster propagation of heat through a body.



Dimensionless Equations:

$$\frac{\partial^2\theta}{\partial X^2} = \frac{\partial\theta}{\partial\tau} \qquad \text{in } 0 < X < 1, \ \text{for } \tau > 0$$

$$\frac{\partial\theta}{\partial X} = 0 \qquad \text{at } X = 0, \ \text{for } \tau > 0$$

$$\frac{\partial\theta}{\partial X} + \text{Bi}\,\theta = 0 \qquad \text{at } X = 1, \ \text{for } \tau > 0$$

$$\theta = 1 \qquad \text{in } 0 \leq X \leq 1, \ \text{for } \tau = 0$$

For plane wall, the solution involves several parameters:

$$T = T(x, L, k, h, \alpha, T_i, T_\infty, t)$$

By using dimensional groups, we can reduce the number of parameters.

$$\theta = \theta(X, \text{Bi}, \text{Fo})$$

The solution for temperature will now be function of the dimensionless quantities:

$$\theta = \theta(X, \mathrm{Bi}, \mathrm{Fo})$$

The transient temperature charts shown in next slides for a large plane wall (also available for long cylinder, and sphere) were presented by M. P. Heisler in 1947 and are called Heisler charts. There are three charts associated with each geometry:

1. The first chart is to determine the temperature T_0 at the center of the geometry at a given time t.
2. The second chart is to determine the temperature at other locations at the same time in terms of T_0.
3. The third chart is to determine the total amount of heat transfer up to the time t.

These plots are valid for $\mathrm{Fo} > 0.2$.

Assumptions:

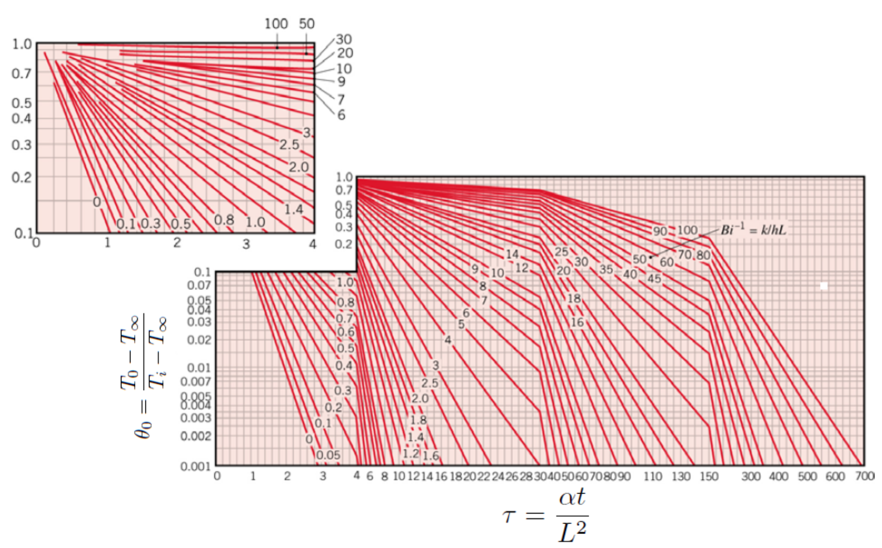
- Uniform initial temperature (T_i) over the entire body.
- Constant T_∞, step change in temperature.
- Simple geometry: slab, cylinder, and sphere.

Limitations:

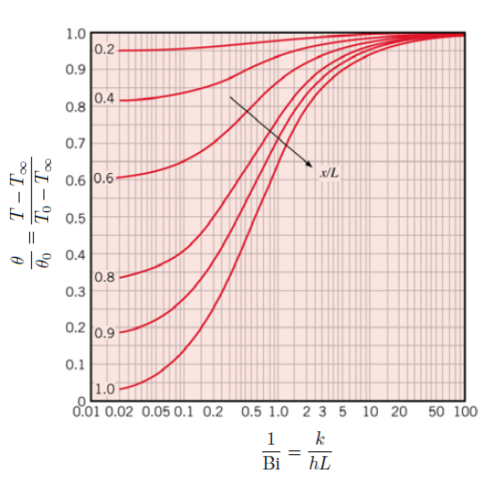
- No heat generation.
- For Fo > 0.2.

Heisler Charts (contd..)

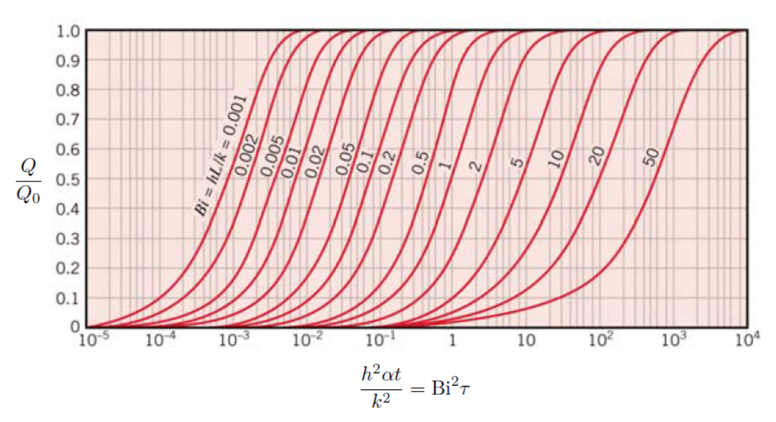
Mid Plane Temperature:



Temperature Distribution:



Change in Thermal Energy Storage:



$$Q_0 = \rho V C_P (T_i - T_\infty)$$

Example 1: Transient Heat Transfer in Cylinder

A 2 m long 0.2 m diameter steel cylinder ($k = 40$ W/m.K, $\alpha = 1 \times 10^{-5}$ m^2/s, $\rho = 7854$ kg/m^3, $C_P = 434$ J/kg.K), initially at 400°C, is suddenly immersed in water at 50°C for quenching process. If the convection coefficient is 200 W/m^2.K, calculate after 20 minutes:

(a) the center temperature

(b) the surface temperature

(c) the heat transfer to the water

Solution:

$L/D = 2/0.2 = 10$; we assume infinitely long cylinder.

Check Lumped Capacitance Method (LCM):

$$\text{Bi} = \frac{hL}{k} = \frac{h(V/A)}{k} = \frac{h(r_o/2)}{k} = \frac{200 \times (0.1/2)}{40} = 0.25$$

Since Bi > 0.25, we can not use LCM, instead we can use Heisler charts.

From the definition of Bi as given in Heisler chart,

$$Bi = \frac{hr_o}{k} = \frac{200 \times 0.1}{40} = 0.5 \qquad \Longrightarrow \qquad \frac{1}{\text{Bi}} = \frac{1}{0.5} = 2$$

and,

$$\text{Fo} = \tau = \frac{\alpha t}{r_o^2} = \frac{1 \times 10^{-5} \times (20 \times 60)}{0.1^2} = 1.2$$

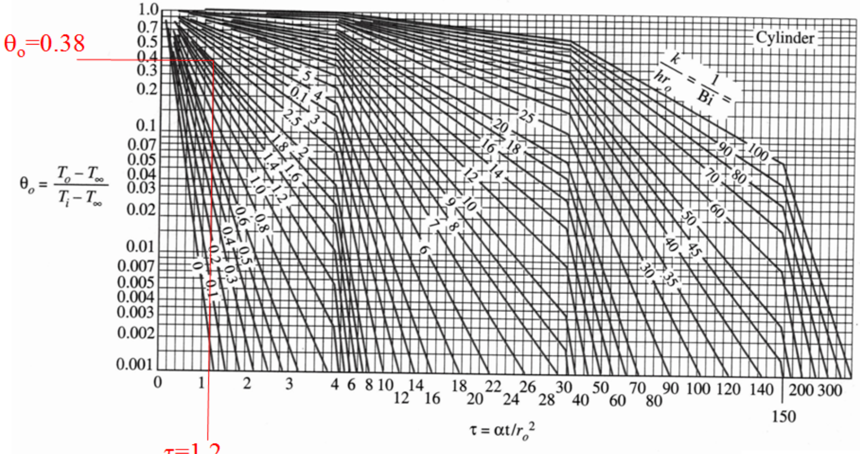
$$\text{Bi}^2\tau = 0.5^2 \times 1.2 = 0.3$$

Heisler Chart Usage - Solved Problem (contd..)

Centreline Temperature (T_0):
For $1/\mathrm{Bi} = 2$, and $\tau = 1.2$, from figure (a), we get $\theta_0 = 0.38$.

$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = 0.38$$

$$\Longrightarrow \quad T_0 = (400 - 50) \times 0.38 + 50 = 183^\circ\mathrm{C}$$



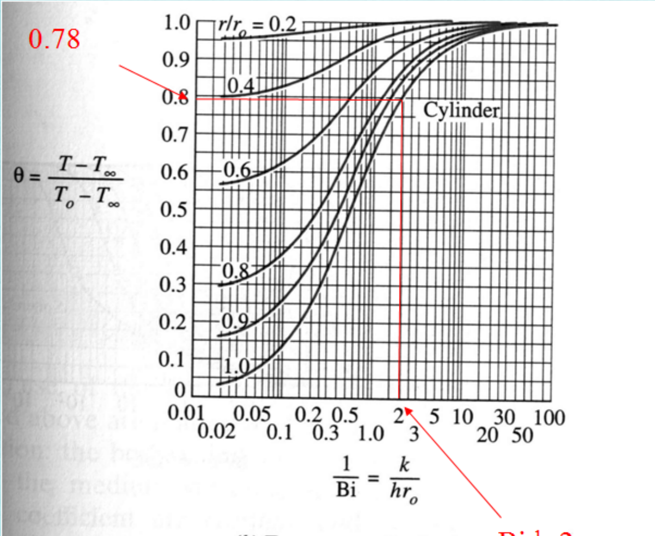
(a) Midplane temperature (from M. P. Heisler)

Surface Temperature (T):

For $r/r_o = 1$, and $1/\mathrm{Bi} = 2$, from figure (b), we get $\theta = 0.78$.

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty} = 0.78$$

$$\implies \quad T = (183 - 50) \times 0.78 + 50 = 153.74^\circ\mathrm{C}$$

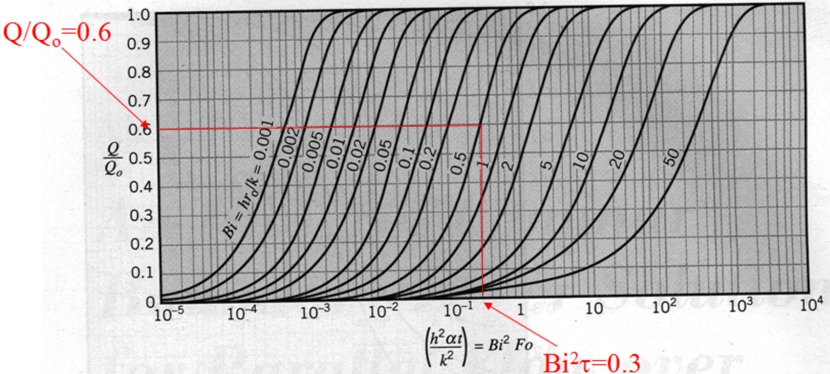


(*b*) Temperature distribution
(from M. P. Heisler)

Heat Transfer (Q):

For $\mathrm{Bi}^2\tau = 0.3$, and $\mathrm{Bi} = 0.5$, from figure (c), we get $Q/Q_0 = 0.6$.

$$\begin{aligned} Q_0 &= \rho V C_P (T_i - T_\infty) \\ &= 7854 \times (\pi \times 0.1^2 \times 2) \times 434 \times (400 - 50) \\ &= 7.5 \times 10^7 \text{ J} \\ Q &= Q_0 \times 0.6 = 4.5 \times 10^7 \text{ J} \end{aligned}$$



Internal energy change as a function of time for an infinite cylinder of radius r_o